PMM U.S.S.R.,Vol.49,No.2,pp.261-264,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

NON-LINEAR PROBLEMS OF CONNECTING COMPOSITE SPATIAL BODIES AND THIN SHELLS, AND VARIATIONAL METHODS FOR THEIR SOLUTION*

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The contact formulation of geometrically non-linear problems of connecting composite spatial bodies, as well as thin composite shells interconnected by a butt (rigidly or not, as a hinge, say) is considered. In this formulation, the composite body (shell) is separated into individual elements, appropriate interaction reactions are introduced into the consideration on the common interface, and an appropriate boundary value problem is formulated for each element. An artificial increase in the number of unknowns of the problem here results in a corresponding increase, in the number of equations because of the replacement of the static connection conditions for the elements by double the number of static boundary conditions on the common interface. While solving the problem, the interaction reactions are determined from the kinematic conditions for the element connections.

It is shown that the advantage of such a formulation for problems of the connection of composite bodies is the considerable simplification of the application of methods of the direct calculus of variations for their solution. To this end, functionals referred to the class of generalized Lagrange functions for problems defined on discontinuous stress, deformation, and displacement fields /1-3/, to which the unknown interaction reactions are also referred together with the displacements to the number of functional arguments, are constructed for spatial composite bodies and composite shells. It is proved that the conditions for their stationarity yield variational equations from which static boundary conditions equivalent to the static connection conditions, and kinematic connection conditions follow in addition to the equilibrium equations and the static boundary conditions on the boundaries where the external static forces are given. According to these equations, the use of direct methods for solving the problems does not require the construction of coordinate functions for the displacements with preliminary satisfaction of the kinematic connection conditions.

1. Differential and variational formulation of non-linear contact problems of the theory of elasticity of composite bodies. A composite body is considered that consists of two elements with volumes $V_{(n)}$ $(n = 1, 2)$ and is in equilibrium under the effect of a certain system of given surface forces $\mathbf{P}_{(n)}$ and volume forces $\mathbf{F}_{(n)}$. It is assumed that the spaces $V_{(n)}$ are parametrized by the curvilinear coordinates $x_{(n)}^{\alpha}$ $(\alpha = 1, 2, 3)$ with the radius-vectors prior to the deformation $\boldsymbol{\rho}_{(n)} = \boldsymbol{\rho}_{(n)}(x_{(n)}^{\alpha})$. The deformations are considered to be small, the displacements are finite, and the fundamental notation is traditional (see /4/, for instance).

Two variations in the formulation of mechanics problems are possible for the body mentioned.

The first is natural and is the formulation of equilibrium equations for each element of the body $(\boldsymbol{\rho}_{(n)}^{*} = \boldsymbol{\rho}_{(n)} + \mathbf{U}^{(n)}$; $\sigma_{(n)}^{\alpha\beta}$ are the stress tensor components)

$$\nabla_{\alpha}^{(n)}(\sigma_{(n)}^{\alpha\beta}\boldsymbol{\rho}_{\beta}^{(n)*}) - \mathbf{F}_{(n)} = 0 \quad (x_{(n)}^{\alpha} \in V_{(n)}) \tag{1.1}$$

the corresponding static and kinematic boundary conditions ($\nu_{\alpha}^{(n)}$ are vector components of the unit normal $\boldsymbol{\nu}_{(n)}$ to $S_{(n)}$ with respect to the basis vectors $\boldsymbol{\rho}_{\alpha}^{(n)}$)

$$\sigma_{(n)}^{\alpha\beta}\boldsymbol{\rho}_{\beta}^{(n)*}\nu_{\alpha}^{(n)} - \mathbf{P}_{(n)} = 0 \quad (x_{(n)}^{\alpha} \in S_{(n)}^{p}) \tag{1.2}$$

$$\mathbf{U}^{(n)} = \mathbf{U}^{(n)s} \quad (x_{(n)}^{\alpha} \in S_{(n)}^{u}) \tag{1.3}$$

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*Prikl.Matem.Mekhan.,49,2,340-344,1985

at those sections $S^{p}_{(n)}$ and $S^{u}_{(n)}$ of the boundary surfaces $S_{(n)}$ on which the surface force $\mathbf{P}_{(n)}$ and displacements $\mathbf{U}^{(n)}$ vectors are given. In addition to (1.2) and (1.3), the static and and kinematic matching conditions

$$\sigma^{\alpha\beta}_{(1)}\boldsymbol{\rho}^{(1)}_{\beta}\nu^{(1)}_{\alpha} = -\sigma^{\alpha\beta}_{(2)}\boldsymbol{\rho}^{(2)}_{\beta}\nu^{(2)}_{\alpha} \quad (x^{\alpha}_{(n)} \in S^{q}) \tag{1.4}$$

$$\mathbf{U}^{(1)} = \mathbf{U}^{(2)} \quad (x^{\alpha}_{(n)} \in S^{q}) \tag{1.5}$$

should be satisfied at points of the common surface connection elements $S^{q}_{(n)} = S^{q}$.

The second formulation is artificial and corresponds to the contact formulation of the problem according to which unknown reactive interaction forces $\mathbf{q}_{(1)} = -\mathbf{q}_{(2)} = \mathbf{q}$ are introduced into consideration on S^{q}. Conditions (1.4) are here replaced by twice the number of static boundary conditions of the form

$$\sigma^{\alpha\beta}_{(1)}\boldsymbol{\rho}^{(1)}_{\beta}\nu^{(1)}_{\alpha} = \mathbf{q}, \quad \sigma^{\alpha\beta}_{(2)}\boldsymbol{\rho}^{(2)}_{\beta}\nu^{(2)}_{\alpha} = -\mathbf{q} \quad (x^{\alpha}_{(n)} \in S^{q}). \tag{1.6}$$

The reactive force vector occurring here is determined when solving the problem using condition (1.5).

We introduce the functional

$$I = \sum_{n=1}^{2}\Big[\iint\limits_{S^{p}_{(n)}} \mathbf{P}_{(n)}\mathbf{U}^{(n)}dS_{(n)} - \iint\limits_{S^{q}} \mathbf{q}_{(n)}\mathbf{U}^{(n)}dS_{(n)} - \iiint\limits_{V_{(n)}} (\mathbf{F}_{(n)}\mathbf{U}^{(n)} - W_{(n)})\, dV_{(n)}\Big] \tag{1.7}$$

into the consideration, which is the total potential energy of the system of elements of the composite body in which $W_{(n)}$ is the specific strain potential energy of the n-th element. This functional differs from the traditional Lagrange functional corresponding to the relationships (1.1),(1.2),(1.4) by the second component that expresses the work of the non-equilibrated reactive force $\mathbf{q}$ on the appropriate displacements. It is easy to prove that the necessary set of relationships corresponding to the second formulation of the problem will follow from the variational equation ($\varepsilon^{(n)}_{\alpha\beta}$ are the contravariant strain tensor components of the n-th element)

$$\delta I = \sum_{n=1}^{2}\Big[\iint\limits_{S^{p}_{(n)}} \mathbf{P}_{(n)}\delta\mathbf{U}^{(n)}dS_{(n)} - \iint\limits_{S^{q}_{(n)}} \mathbf{q}_{(n)}\delta\mathbf{U}^{(n)}dS_{(n)} + \iiint\limits_{V_{(n)}} (\mathbf{F}_{(n)}\delta\mathbf{U}^{(n)} - \sigma^{\alpha\beta}_{(n)}\delta\varepsilon^{(n)}_{\alpha\beta})\, dV_{(n)}\Big] + \iint\limits_{S^{q}} (\mathbf{U}^{(1)} - \mathbf{U}^{(2)})\,\delta\mathbf{q}\, dS^{q} = 0 \tag{1.8}$$

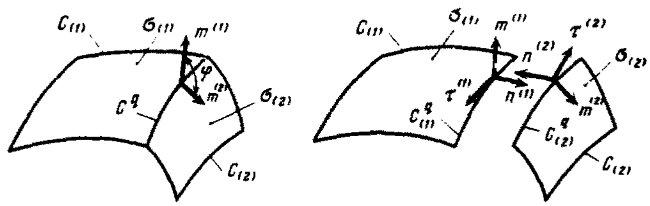
if the kinematic relationships

$$2\varepsilon^{(n)}_{\alpha\beta} = \nabla^{(n)}_{\alpha}\mathbf{U}^{(n)}\boldsymbol{\rho}^{(n)}_{\beta} + \nabla^{(n)}_{\beta}\mathbf{U}^{(n)}\boldsymbol{\rho}^{(n)}_{\alpha} + \nabla^{(n)}_{\alpha}\mathbf{U}^{(n)}\nabla^{(n)}_{\beta}\mathbf{U}^{(n)}$$

are introduced here and the traditional transformations are performed using the Gauss's theorem. A feature of the application of (1.8) to solve problems by direct methods is the need to construct coordinate functions for the two-dimensional unknowns $q^{\alpha}_{(n)} = \mathbf{q}_{(n)}\boldsymbol{\rho}^{\alpha}_{(n)}$ and the three-dimensional unknowns $U^{(n)}_{\alpha} = \mathbf{U}^{(n)}\boldsymbol{\rho}^{(n)}_{\alpha}$, but without preliminary satisfaction of conditions (1.5) and (1.6).

2. Variational formulation of non-linear contact problems for thin composite shells. Without loss of generality we shall consider a structure consisting of two shells with middle surfaces $\sigma_{(n)}$ and boundary contours $C_{(n)} \in \sigma_{(n)}$ which are connected without eccentricity by a junction at a certain common section $C^{q}_{(n)} = C^{q} \in C_{(n)}$. We refer the shell spaces $V_{(n)}$ to curvilinear coordinate systems $x^{i}_{(n)}$, $x^{3}_{(n)} = z_{(n)}$ related normally to the surfaces $\sigma_{(n)}$ by the equations $\boldsymbol{\rho}_{(n)} = \mathbf{r}_{(n)} + z_{(n)}\mathbf{m}_{(n)}$ in which $\mathbf{r}_{(n)} = \mathbf{r}_{(n)}(x^{i}_{(n)})$ are radii-vectors of the points $M_{(n)} \in \sigma_{(n)}$, and $\mathbf{m}_{(n)}$ are unit vectors normal to $\sigma_{(n)}$. Here $\mathbf{r}^{(n)}_{i}$ are coordinate vectors on $\sigma_{(n)}$, $a^{(n)}_{ik} = \mathbf{r}^{(n)}_{i}\mathbf{r}^{(n)}_{k}$, $b^{(n)}_{ik} = -\mathbf{m}^{(n)}_{i}\mathbf{r}^{(n)}_{k} = -\mathbf{r}^{(n)}_{k}\partial\mathbf{m}_{(n)}/\partial x^{i}_{(n)}$ are coefficients of the first and second quadratic forms of the surface $\sigma_{(n)}$, $\mathbf{n}_{(n)}$ and $\boldsymbol{\tau}_{(n)}$ are unit vectors of the normal and tangent to the lines $C_{(n)}$ in a plane tangent to $\sigma_{(n)}$, $\nabla^{(n)}_{i}$ are the symbols of the covariant derivatives relative to $a^{(n)}_{ik}$, and φ is the angle between the vectors $\mathbf{m}_{(1)}$ and $\mathbf{m}_{(2)}$ at points of the section $C^{q} \in C_{(n)}$ at which, according to the figure, the following relations hold

$$\mathbf{n}_{(2)} = -\cos\varphi\,\mathbf{n}_{(1)} + \sin\varphi\,\mathbf{m}_{(1)}, \quad \mathbf{m}_{(2)} = \sin\varphi\,\mathbf{n}_{(1)} + \cos\varphi\,\mathbf{m}_{(1)}, \quad \boldsymbol{\tau}_{(1)} = -\boldsymbol{\tau}_{(2)}.$$



Within the framework of the hypotheses of the classical Kirchhoff-Love theory, the vectors of the finite displacements $U^z_{(n)}$ and the covariant tangential strain tensor components $\varepsilon^{z(n)}_{ik}$ at the points $(x^i_{(n)}, z_{(n)})$ of the shell are given by /4/

$$U^z_{(n)} = U^{(n)} = v^{(n)} + z_{(n)}\,(\mathbf{m}^{(n)}_* - \mathbf{m}_{(n)}), \quad -h_{(n)}/2 \leqslant z_{(n)} \leqslant h_{(n)}/2 \tag{2.1}$$

$$\varepsilon^{z(n)}_{ik} = \varepsilon^{(n)}_{ik} + z_{(n)}\varkappa^{(n)}_{ik} \tag{2.2}$$

$$2\varepsilon^{(n)}_{ik} = \mathbf{r}^{(n)}_i \nabla^{(n)}_k \mathbf{v}^{(n)} + \mathbf{r}^{(n)}_k \nabla^{(n)}_i \mathbf{v}^{(n)} + \nabla^{(n)}_i \mathbf{v}^{(n)} \nabla^{(n)}_k \mathbf{v}^{(n)} \tag{2.3}$$

$$2\varkappa^{(n)}_{ik} = b^{(n)}_{ik} - b^{(n)*}_{ik} = b^{(n)}_{ik} + \mathbf{r}^{(n)*}_i \mathbf{m}^{(n)*}_k + \mathbf{r}^{(n)*}_k \mathbf{m}^{(n)*}_i \tag{2.4}$$

where $\mathbf{r}^{(n)}_* = \mathbf{r}^{(n)} + \mathbf{v}^{(n)}$ and $\mathbf{r}^{(n)*}_i$ are the radius-vectors of points of the strained middle surfaces $\sigma^*_{(n)}$ and the coordinate vectors thereon, $\mathbf{v}^{(n)}$ are displacement vectors of the points $\sigma_{(n)}$, $\mathbf{m}^*_{(n)}$, $\mathbf{m}^{(n)*}_i$ are unit vectors of the normals to $\sigma^*_{(n)}$ and their partial derivatives with respect to $x^i_{(n)}$.

By using (2.1), the volume forces $F_{(n)}$ and the surface forces acting on the boundary surfaces $z_{(n)} = \pm h_{(n)}/2$ are reduced /4/ to surface force and moment vectors

$$\mathbf{X}_{(n)} = X^i_{(n)}\mathbf{r}^{(n)*}_i + X^3_{(n)}\mathbf{m}^*_{(n)}, \quad \mathbf{M}_{(n)} = M^i_{(n)}\mathbf{r}^{(n)*}_i \tag{2.5}$$

referred to the unit areas $\sigma_{(n)}$, while the external surface forces acting on the boundary cuts are reduced to contour force and moment vectors

$$\mathbf{\Phi}^s_{(n)} = \Phi^{(n)}_n \mathbf{n}^*_{(n)} + \Phi^{(n)}_{n\tau}\boldsymbol{\tau}^*_{(n)} + \Phi^{(n)}_m \mathbf{m}^*_{(n)}, \quad \mathbf{M}^{(n)}_n = L^{(n)}_{n\tau}\mathbf{n}^*_{(n)} + L^{(n)}_n \boldsymbol{\tau}^*_{(n)} \tag{2.6}$$

acting on the sections $C^p_{(n)} \in C_{(n)}$ which are referred to unit length $C_{(n)}$ and are expanded over the appropriate axes of the deformed trihedra $\{\mathbf{n}^*_{(n)}, \boldsymbol{\tau}^*_{(n)}, \mathbf{m}^*_{(n)}\}$.

Using this approach, let us separate the construction into two shells and, by analogy with (2.6), reduce the reactor force vector $\mathbf{q} = \mathbf{q}_{(1)}$ on the common contact surface S^q to the principal vector $\mathbf{Q} = \mathbf{Q}_{(1)}$ and the principle moment $\mathbf{R} = \mathbf{R}_{(1)}$

$$\mathbf{Q} = Q_n \mathbf{n}^*_{(1)} + Q_{n\tau}\boldsymbol{\tau}^*_{(1)} + Q_m \mathbf{m}^*_{(1)}, \quad \mathbf{R} = R_{n\tau}\mathbf{n}^*_{(1)} + R_n \boldsymbol{\tau}^*_{(1)}$$

referred to unit length of the separating line C^q.

By the assumptions made for the shell system under consideration, the variational equation (1.8) can be reduced to the form

$$\begin{aligned}
\delta I = \sum_{n=1}^{2} \Big[\int\limits_{C^p_{(n)}} (\mathbf{\Phi}^s_{(n)}\delta\mathbf{v}^{(n)} + \mathbf{L}^s_{(n)}\delta\mathbf{m}^*_{(n)})\, dC_{(n)} + {} \\
\int\limits_{C^q_{(n)}} (\mathbf{Q}_{(n)}\delta\mathbf{v}^{(n)} + \mathbf{H}_{(n)}\delta\mathbf{m}^*_{(n)})\, dC_{(n)} + \iint\limits_{\sigma_{(n)}} (\mathbf{X}_{(n)}\delta\mathbf{v}^{(n)} + \mathbf{M}_{(n)}\delta\mathbf{m}^*_{(n)} - {} \\
T^{ik}_{(n)}\delta\varepsilon^{(n)}_{ik} - M^{ik}_{(n)}\delta\varkappa^{(n)}_{ik})\, d\sigma_{(n)} + \int\limits_{C^q} (\mathbf{m}^*_{(n)} - \mathbf{m}_{(n)})\,\delta\mathbf{H}_{(n)}\, dC_{(n)} \Big] + {} \\
\int\limits_{C^q_{(1)}} (\mathbf{v}^{(1)} - \mathbf{v}^{(2)})\,\delta\mathbf{Q}\, dC_{(1)} = 0
\end{aligned} \tag{2.8}$$

$$\mathbf{Q}_{(2)} = -\mathbf{Q}, \quad \mathbf{L}^s_{(n)} = L^{(n)}_n \mathbf{n}^*_{(n)} - L^{(n)}_{n\tau}\boldsymbol{\tau}^*_{(n)}, \quad \mathbf{H}_{(n)} = R^{(n)}_n \mathbf{n}^*_{(n)} - R^{(n)}_{n\tau}\boldsymbol{\tau}^*_{(n)}$$

$$\mathbf{H}_{(2)} = -\mathbf{H}_{(1)}$$

in which $T^{ik}_{(n)}$, $M^{ik}_{(n)}$ are the contravariant components of the internal tangential force and bending moment tensors, where by virtue of $\boldsymbol{\tau}^*_{(1)} = -\boldsymbol{\tau}^*_{(2)}$ and $\mathbf{Q}_{(2)} = -\mathbf{Q}, \mathbf{R}_{(2)} = -\mathbf{R}$ the equations $Q^{(2)}_{n\tau} = Q_{n\tau}$, $R^{(2)}_n =$

R_n hold on C^q.

Equation (2.8) obtained in a scalar representation is used to solve contact shell interaction problems by Ritz's method in the case of their rigid connection on the junction line, according to which five one-dimensional coordinate functions are constructed for the components of the vectors $\mathbf{Q}$, $\mathbf{R}$ and six two-dimensional coordinate functions are constructed for the displacement vector components $\mathbf{v}^{(n)}$ without first satisfying the kinematic shell connection conditions

$$\mathbf{v}^{(1)} = \mathbf{v}^{(2)},\ [\mathbf{m}^*_{(1)}, \mathbf{m}^*_{(1)} - \mathbf{m}_{(1)}] = [\mathbf{m}^*_{(2)}, \mathbf{m}^*_{(2)} - \mathbf{m}_{(2)}] \quad (x^i_{(n)} \in C^q)\ . \tag{2.9}$$

In the case of a hinged connection of the shells, it is sufficient to set $R_n = 0$ in (2.8), which results in reducing the number of desired unknowns in the problem by one.

If relationships (2.3) and (2.4) are substituted into (2.8) and the usual manipulations /4/ are performed, it is possible to arrive at another form of the variational equation constructed, which is a vector inscription of the equation for the Bubnov-Galerkin method.

Remark. The variational problem for the functional (1.8) can be obtained from the variational problem for the Lagrange functional of a composite body

$$I_1 = \sum_{n=1}^{2} \Big[\iint\limits_{S^p_{(n)}} \mathbf{P}_{(n)} \mathbf{U}^{(n)} dS_{(n)} + \iiint\limits_{V_{(n)}} (\mathbf{F}_{(n)} \mathbf{U}^{(n)} - W_{(n)})\, dV_{(n)} \Big]$$

under the constraint (1.5) using the well-known method /2, 5/ of introducing the undetermined Lagrange multiplier, whose mechanical analogue is indeed the reactive interaction force vector $\mathbf{q}$. Such a method is utilized, in particular, in mixed linear problems with unknown reactions in relations for system of deformable elements with a finite number of degrees of freedom /2/.

We also note that within the framework of the relationships of the classical Kirchhoff-Love theory of shells there is the possibility of satisfying just four scalar connection conditions

$$\mathbf{v}^{(1)} = \mathbf{v}^{(2)},\quad \mathbf{n}_{(1)} \mathbf{m}^*_{(1)} = \mathbf{n}_{(2)} \mathbf{m}^*_{(2)} \quad (x^i_{(n)} \in C^q)\ . \tag{2.10}$$

Consequently, manipulation by using the method of Lagrange undetermined multipliers of the ordinary Lagrange functional written for a composite shell under the additional conditions (2.10) gives a result different from that obtained.

The range of application of this approach is not limited just to problems of the connection of composite bodies and thin shells. Effective direct methods of solving three-dimensional problems of the theory of elasticity and two-dimensional problems of shell theory in non-canonical domains can be developed on its basis.

REFERENCES

1. PRAGER W., Variational principles of the linear static theory of elasticity for discontinuous displacements, strains, and stresses. Mekhanika /Russian translations/, No.5, 1969.
2. ROZIN L.A., Variational Formulations of Problems for Elastic Systems. Izdat. Leningrad. Gosudarst. Univ., Leningrad, 1978.
3. ABOVSKII N.P., ANDREYEV N.P., and DERUGA A.P., Variational Principles of the Theory of Elasticity and the Theory of Shells. Nauka, Moscow, 1978.
4. GALIMOV K.Z., Principles of the Non-linear Theory of Thin Shells, Izdat. Kazan. Univ., Kazan', 1975.
5. ECKLUND I. and TEMAM R., Convex Analysis and Variational Problems, Mir, Moscow, 1979.

Translated by M.D.F.